End effects in inviscid flow in a magnetohydrodynamic channel

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(Received 16 January 1961)

The flow of an inviscid, incompressible electrical conducting fluid in a channel of constant rectangular cross-section is considered, when the flow enters a region which contains a magnetic field transverse to the flow and electrodes on opposite sides of the channel. This geometry is typical of a d.c. induction pump or magnetohydrodynamic generator. The conducting fluid external to the magnetic field acts as a shunt and produces a non-uniform electric potential field and hence a non-uniform Lorenz force on the fluid, and causes the fluid velocity profile to be distorted. These effects are calculated theoretically for small magnetic Reynolds number and small magnetic interaction parameter. It is found that the velocity at the centre-line of the channel is retarded and at the walls the velocity is accelerated. The fractional change of velocity at the wall is equal to approximately 0·44 times a modified magnetic interaction parameter.

1. Introduction

In a magnetohydrodynamic electrical power generator, an electrically conducting fluid flows through a channel, in which a transverse magnetic field is applied. The magnetic field interacts with the fluid flow to induce an e.m.f. in the direction perpendicular to the magnetic field and to the direction of the fluid flow. Electrodes are inserted into the channel walls parallel to the magnetic field. If the electrodes are connected to an external load, electrical power can be extracted from the magnetohydrodynamic generator, since the induced e.m.f. produces a voltage difference between the electrodes (figure 1). Alternatively, if the electrodes are connected to a current supply, the channel will act as an induction pump.

The electric current in the fluid is composed of two parts: one due to the induced e.m.f., and the other due to the electric field caused by the voltage difference between the two electrodes. The latter current flows from the positive to the negative electrode and opposes the current caused by the induced e.m.f. (figure 2). At the entrance and exit regions of the channel, that is, near the ends of the electrodes, electric field is not uniform, causing the current density to vary in magnitude and direction in the entrance and exit regions.

The Lorentz force due to the induced e.m.f. opposes the flow and is constant, but the Lorentz force due to the voltage differences between the electrodes

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varies in magnitude and direction in the entrance and exit regions. The variation of the Lorentz force causes the velocity profile of the fluid to change in the en-

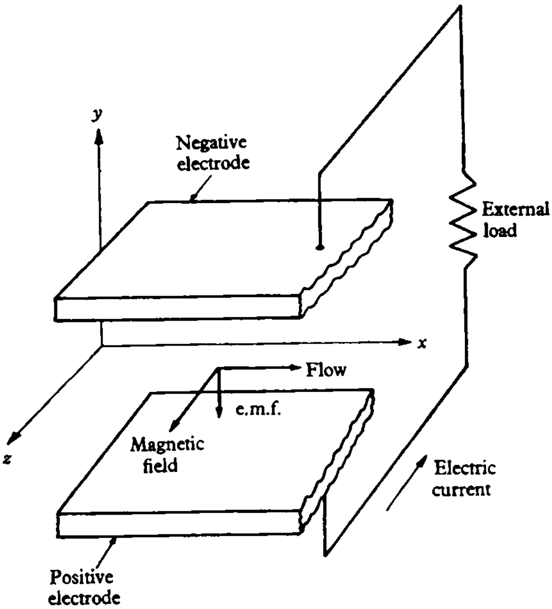

FIGURE 1. Schematic diagram of a magnetohydrodynamic channel.

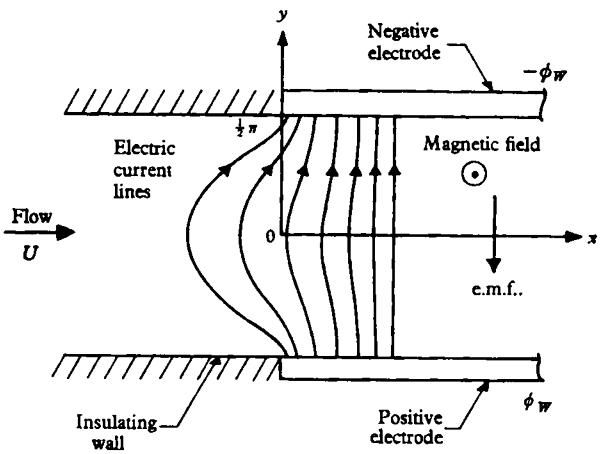

FIGURE 2. Electric current lines due to electric fields in the entrance region of the channel.

trance region and then to assume an asymptotic shape throughout the channel. The Lorentz forces in the exit region will be identical to the Lorentz forces in the entrance region, and will cause further distortion to the flow.

In this investigation, the fluid dynamic effects of the Lorentz forces in the entrance region are determined. Analytical expressions are found which show the manner in which the fluid velocity is altered by the Lorentz forces. The results are illustrated by a set of velocity profiles and a diagram of the streamline pattern.

In order to obtain a solution to the problem, several simplifying assumptions and idealized boundary conditions have been used. The fluid dynamic assumptions are as follows: (1) The flow is steady. (2) The density of the fluid is constant. (3) The viscosity is negligible. (4) The flow is two-dimensional; that is, all gradients in the direction of the magnetic field are zero. (5) The velocity perturbations are small, hence the flow equations may be linearized.

The magnetohydrodynamic assumptions are as follows: (1) The electrical conductivity of the fluid is constant. (2) The Hall effects are negligible. (3) The magnetic Reynolds number is assumed very small so that the applied magnetic field is not altered by the presence of electric currents in the fluid. (4) The magnetic interaction parameter is assumed to be small.

The idealized boundary conditions are as follows: (1) The channel walls, including the electrodes, are parallel. (2) The electrodes are infinitely long, $0 < x < \infty$. (3) The voltage of each electrode is constant, $\pm \phi_W$. (4) The walls, except for the electrodes, are perfect electrical insulators. (5) The magnetic field is zero upstream from the electrodes and has a constant value between the electrodes. It is directed perpendicular to the diagram in figure 2. (6) The velocity far upstream from the electrodes is constant across the channel.

2. Equations

Since the velocity perturbations are assumed small, the axial and transverse velocities can be written as

$$u = U + u', \quad v = v', \tag{1}$$

where u' and v' are small compared to U. Thus, the fluid dynamic equations may be linearized as follows, the continuity equation becoming

$$(\partial u'/\partial x) + (\partial v'/\partial y) = 0, \tag{2}$$

and the momentum equations becoming

$$\rho U(\partial u'/\partial x) + (\partial p/\partial x) = j_y B, \tag{3}$$

and

$$\rho U(\partial v'/\partial x) + (\partial p/\partial y) = -j_x B. \tag{4}$$

Differentiation of equation (3) with respect to y and equation (4) with respect to x and subtraction of the resulting equations gives

$$\rho U(\partial \zeta/\partial x) = -[j_y(\partial B/\partial y) + B(\partial j_y/\partial y) + j_x(\partial B/\partial x) + B(\partial j_x/\partial x)], \tag{5}$$

where the vorticity ζ is given by

$$\zeta = (\partial v'/\partial x) - (\partial u'/\partial x). \tag{6}$$

The first term on the right of equation (5) is zero. From the continuity equation of electricity, the second and fourth terms on the right are also zero. Thus equation (5) becomes

$$\rho U(\partial \zeta/\partial x) = -j_x(\partial B/\partial x). \tag{7}$$

The magnetic field, B, is zero or constant everywhere except at $x = 0$. Therefore, for $x < 0$, the vorticity is a function of y alone and for $x > 0$, it is another function of y alone. Since the flow is initially irrotational, it remains irrotational up to $x = 0$. At $x = 0$, there is a vorticity jump which is given by

$$\rho U \Delta\zeta = -(j_x)_{x=0} \Delta B, \tag{8}$$

where $(j_x)_{x=0}$ is a function of y only. Thus

$$\left.\begin{aligned} \zeta &= 0 && \text{for} \quad x < 0, \\ \text{and} \qquad \zeta &= -(B/\rho U)(j_x)_{x=0} && \text{for} \quad x > 0. \end{aligned}\right\} \tag{9}$$

If the longitudinal component of the electric current density at $x = 0$ is known, the vorticity can be found anywhere in the channel. The vorticity may be expressed in terms of the stream function by

$$u' = \partial\psi/\partial y, \quad v' = -\partial\psi/\partial x. \tag{10}$$

The vorticity is then given by

$$\zeta = -(\partial^2\psi/\partial x^2) - (\partial^2\psi/\partial y^2) = -\nabla^2\psi,$$

so that equation (9) becomes

$$\left.\begin{aligned} \nabla^2\psi &= 0 && (x < 0); \\ \nabla^2\psi &= (B/\rho U)(j_x)_{x=0} && (x > 0). \end{aligned}\right\} \tag{11}$$

Equation (11) is the required fluid dynamical equation. The electric field equation is obtained from conservation of electricity. Neglecting the Hall effect, the generalized Ohm's law is taken as

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{q} \times \mathbf{B}). \tag{12}$$

Since the problem is steady, $\nabla \times \mathbf{E} = 0$, and therefore $\mathbf{E}$ may be considered to be the negative gradient of electric potential

$$\mathbf{E} = -\nabla\phi. \tag{13}$$

The divergence of the electric current is zero; for constant σ equation (12) then becomes

$$\nabla^2\phi = \mathbf{B}.(\nabla \times \mathbf{q}) - \mathbf{q}.(\nabla \times \mathbf{B}). \tag{14}$$

Since $\nabla \times \mathbf{q} = \zeta\mathbf{k}$, $\mathbf{B} = B\mathbf{k}$, equation (14) becomes

$$\nabla^2\phi = B\zeta - v(\partial B/\partial x). \tag{15}$$

For $x \neq 0$, $(\partial B/\partial x) = 0$, and the vorticity is obtained from equation (9), so that equation (15) becomes

$$\left.\begin{aligned} \nabla^2\phi &= 0 && (x < 0); \\ \nabla^2\phi &= -(B^2/\rho U)(j_x)_{x=0} && (x > 0). \end{aligned}\right\} \tag{16}$$

The electric current is obtained from equation (12) as

$$j_x = \sigma(E_x + vB) \tag{17}$$

or

$$j_x = -\sigma[(\partial\phi/\partial x) + B(\partial\psi/\partial x)]. \tag{18}$$

Thus equations (11) and (16) become

$$\nabla^2\phi = \nabla^2\psi = 0 \quad (x < 0); \tag{19a}$$

$$\nabla^2\psi = -\frac{\sigma B}{\rho U}\left(\frac{\partial\phi}{\partial x} + B\frac{\partial\psi}{\partial x}\right)_{x=0} \quad (x > 0); \tag{19b}$$

$$\nabla^2\phi = \frac{\sigma B^2}{\rho U}\left(\frac{\partial\phi}{\partial x} + B\frac{\partial\psi}{\partial x}\right)_{x=0} \quad (x > 0). \tag{19c}$$

The boundary conditions are

$$\phi = \pm\phi_W, \quad y = \mp\tfrac{1}{2}\pi \quad (x > 0); \tag{20a}$$

$$\partial\phi/\partial y = 0, \quad y = \pm\tfrac{1}{2}\pi \quad (x < 0); \tag{20b}$$

$$\psi = \pm U\tfrac{1}{2}\pi, \quad y = \pm\tfrac{1}{2}\pi \quad (-\infty < x < \infty). \tag{20c}$$

Equations (19) may be made dimensionless by use of the following new variables

$$\psi = Uh\Psi, \quad \phi = UBh\Phi, \quad x = h\xi, \quad y = h\xi', \tag{21}$$

where h is the channel width.

With the use of equations (21), equations (19b) and (19c) become

$$\nabla^2\Phi = I(\partial/\partial\xi)(\Phi + \Psi), \tag{22a}$$

and

$$\nabla^2\Psi = I(\partial/\partial\xi)(\Phi + \Psi), \tag{22b}$$

where I is the magnetic interaction parameter, $\sigma B^2 h/\rho U$. To solve equations (22a) and (22b) for small I, Φ and Ψ are expressed in terms of series whose coefficients are powers of I

$$\Phi = \Phi_0 + I\Phi_1 + \dots, \tag{23a}$$

$$\Psi = \Psi_0 + I\Psi_1 + \dots. \tag{23b}$$

These are then substituted into equations (22a) and (22b), and the terms having common powers of I are equated, to give

$$\nabla^2\Phi_0 = 0 \quad (-\infty < x < \infty); \tag{24a}$$

$$\nabla^2\Psi_0 = 0 \quad (-\infty < x < \infty); \tag{24b}$$

$$\nabla^2\Phi_1 = 0 \quad (x < 0); \tag{24c}$$

$$\nabla^2\Phi_1 = (\partial/\partial\xi)(\Phi_0 + \Psi_0) \quad (x > 0); \tag{24d}$$

$$\nabla^2\Psi_1 = 0 \quad (x < 0); \tag{24e}$$

$$\nabla^2\Psi_1 = (\partial/\partial\xi)(\Phi_0 + \Psi_0) \quad (x > 0). \tag{24f}$$

The boundary conditions then become, in terms of the dimensional variables,

$$\phi_0 = \pm\phi_W, \quad y = \mp\tfrac{1}{2}\pi \quad (x > 0); \tag{25a}$$

$$\partial\phi_0/\partial y = 0, \quad y = \pm\tfrac{1}{2}\pi \quad (x < 0); \tag{25b}$$

$$\phi_1 = 0, \quad y = \pm\tfrac{1}{2}\pi \quad (x > 0); \tag{25c}$$

$$\partial\phi_1/\partial y = 0, \quad y = \pm\tfrac{1}{2}\pi \quad (x < 0); \tag{25d}$$

$$\psi_0 = \pm\tfrac{1}{2}\pi U, \quad y = \pm\tfrac{1}{2}\pi \quad (-\infty < x < \infty); \tag{25e}$$

$$\psi_1 = 0, \quad y = \pm\tfrac{1}{2}\pi \quad (-\infty < x < \infty). \tag{25f}$$

3. Solution to the electric potential equations

It is convenient to make the transformation

$$\eta = y + \tfrac{1}{2}\pi \tag{26}$$

to shift the origin of the co-ordinate system to the lower wall, see figure 2. To solve equation (24*a*) with the boundary conditions of equations (25*a*, *b*), the original region is mapped conformally by means of the following transform (see figure 3),

$$e^{z} = \sin z', \tag{27}$$

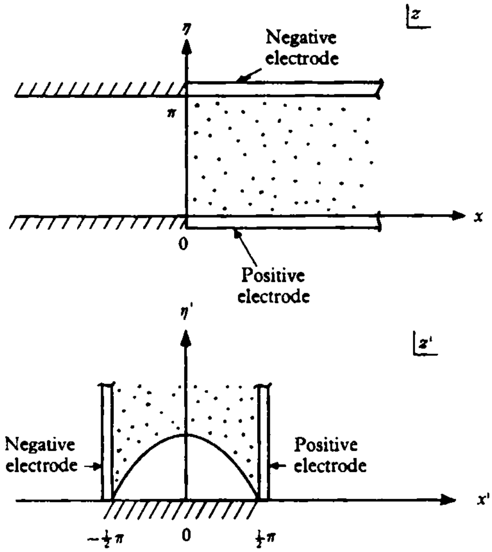

FIGURE 3. Complex planes showing the regions mapped.

where $z = x + i\eta$, $z' = x' + i\eta'$. In the z'-plane, the boundary conditions are

$$\phi_0 = \pm \phi_W, \quad x' = \pm \tfrac{1}{2}\pi; \quad \partial\phi_0/\partial\eta' = 0 \quad (\eta' = 0). \tag{28}$$

The solution is obviously

$$\phi_0 = 2\phi_W x'/\pi. \tag{29}$$

Thus, in the z'-plane lines of constant η' are flux lines, while lines of constant x' are equipotential lines. These are shown in figure 4.

The electric current to the electrodes between $x = 0$ and x may be calculated next from

$$J_0 = \int_0^x j_y\, dx = -\sigma \int_0^x [(\partial\phi_0/\partial y) + UB]\, dx = \sigma\left[\int_0^{\eta'} (\partial\phi_0/\partial x')\, d\eta' - UBx\right]$$

or

$$J_0 = 2\pi^{-1}\sigma\phi_W \eta' - \sigma UBx. \tag{30}$$

For large η' along $x' = \frac{1}{2}\pi$, the transformation of equation (27) becomes $e^{x} = \frac{1}{2}e^{\eta'}$, so that

$$\eta' = x + \ln 2, \tag{31}$$

and when substituted into equation (30) this yields

$$J_0 = \sigma(2\pi^{-1}\phi_W - UB)\, x + 2\pi^{-1}\sigma\phi_W \ln 2. \tag{32}$$

Thus, there is a current loss of $2\pi^{-1}\sigma\phi_W \ln 2$ associated with the inlet, and a similar loss at the exit. If the channel length is L, the total current to the electrodes, per unit height in the direction of the magnetic field, is

$$J_L = \sigma[(2\phi_W/h) - UB]\,L + 4\pi^{-1}\sigma\phi_W \ln 2, \tag{33}$$

where h is the channel width and the voltage across the electrodes is $2\phi_W$; thus the last term represents the inlet and exit losses. The power consumed or produced is $P = -2\phi_W J_L$, and, in the case of a magnetohydrodynamic generator, the external resistance is given by $-2\phi_W/J_L$.

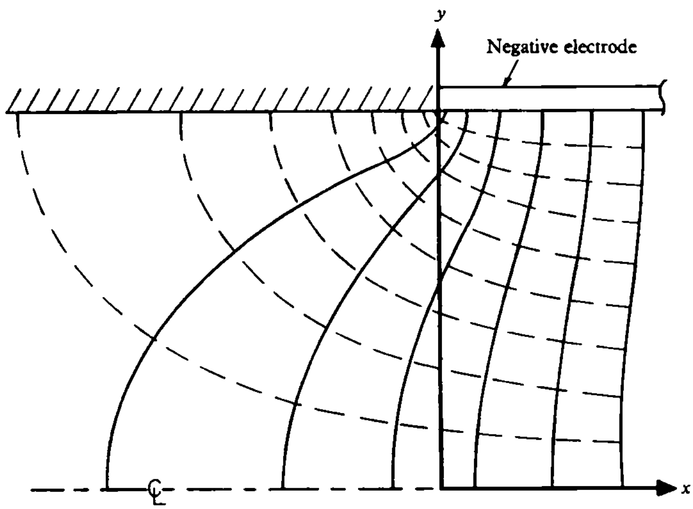

FIGURE 4. Electric flux lines (solid) and equipotential lines (dashed) in the upper half of the channel.

The change in total average pressure may be calculated from

$$\Delta p = 2\pi^{-1}\int_{-\infty}^{\infty} (\mathbf{j}\times\mathbf{B})_x\,dx\,dy. \tag{34}$$

Substitution for the current from the generalized Ohm's laws and integration yields

$$\Delta p = \sigma BL[2\phi_W h^{-1} - UB]. \tag{35}$$

It is interesting to note that there is no term representing the end loss in equation (35).

A pump or generator efficiency, ϵ_p or ϵ_g, may be defined in terms of the flow work $(\Delta p)\,Uh$ and the power P by

$$\epsilon_p = \epsilon_g^{-1} = \frac{\Delta p U h}{P} = \frac{BLUh[2\phi_W h^{-1} - UB]}{2\phi_W\{L[2\phi_W h^{-1} - UB] + 4\pi^{-1}\phi_W \ln 2\}}. \tag{36}$$

Finally, if $L \to \infty$, the end losses become negligible, and the efficiencies become

$$\epsilon_p = \epsilon_g^{-1} = UBh/2\phi_W. \tag{37}$$

Thus for efficient operation, the electrode voltage should be close to the induced voltage. It should be noted that this analysis applies to finite length electrodes with an error of less than 1 % if $L/h > 1$.

The equation for ϕ_0, equation (25a), has also been solved for constant magnetic field extensions by Fishman (1959) and for exponentially decreasing magnetic fields by Sutton, Hurwitz & Poritsky (1959), for which efficiencies have also been calculated. In addition, Hurwitz, Kilb & Sutton (1961) have also considered the Hall effect. However the influence on the fluid motion was not considered. In addition, the electric fields associated with extensions to the magnetic fields complicate the calculation of ψ_1, but do not materially alter the main effect. Therefore, in the remainder, extensions to the magnetic field will not be considered.

The solution to equations (24c, d) for ϕ_1 are also not simple to obtain because of the inhomogeneous term and the boundary conditions for ϕ_1 but this is not required in obtaining the solution for ψ_0 or ψ_1. However, the expression for $\partial\phi_0/\partial x$ is required. In terms of the transformed coordinates, this is given by

$$(\partial\phi_0/\partial x)_{x=0} = 2\pi^{-1}\phi_W(\partial x'/\partial x)_{x=0}. \tag{38}$$

Use of equation (27) to determine (dx'/dx) yields

$$(\partial\phi_0/\partial x)_{x=0} = \pi^{-1}\phi_W[(1\pm\sin\eta)/\sin\eta]^{\frac{1}{2}}. \tag{39}$$

4. Solution to the flow equations

The solution to equation (24b) with boundary conditions (25e) is

$$\psi_0 = Uy. \tag{40}$$

With equations (40), (39) and (24e), the equation for ψ_1 then becomes

$$\nabla^2\psi_1 = f(y), \tag{41}$$

where $f(y) = 0$ when $x < 0$

and $f(y) = -(\sigma B\phi_W/\pi\rho U)[(1-\cos y)/\cos y]^{\frac{1}{2}}$ when $x > 0.$ (42)

Since $f(y)$ is a forcing function, the solution for ψ_1 may be obtained by means of a Green's function as

$$\psi_1 = \int_0^\infty G(x-\xi, y)\, d\xi, \tag{43}$$

where

$$\nabla^2 G_{1,2}(x-\xi, y) = \delta(\xi)f(y) \tag{44}$$

and δ is the Dirac delta function. By means of separation of variables, the following solution is obtained for G,

$$G_1 = \sum_{n=1}^{\infty} a_n \sin(2ny)\, e^{2n(x-\xi)} \quad \text{when} \quad x < \xi, \tag{45a}$$

and

$$G_2 = \sum_{n=1}^{\infty} a_n \sin(2ny)\, e^{-2n(x-\xi)} \quad \text{when} \quad x > \xi. \tag{45b}$$

The coefficients a_n are the same in equations (45a, b) since $G_1 = G_2$ at $x = \xi$. Substitution of equations (45a, b) in equation (44) and integration from $x = \xi - \epsilon$ to $x = \xi + \epsilon$ yields

$$-4\sum_{n=1}^{\infty} na_n \sin(2ny) = f(y), \tag{46}$$

so that

$$a_n = \frac{1}{n\pi}\int_{-\frac{1}{2}\pi}^{0} f(y)\sin(2ny)\, dy. \tag{47}$$

Use of equation (42) for $f(y)$ and a change of variable to η by equation (26) yields

$$a_n = \frac{(-1)^n}{n\pi}\left(\frac{B\phi_W}{\pi\rho U}\right)\int_0^{\frac{1}{2}\pi}\sin(2n\eta)\left[\frac{1-\sin\eta}{\sin\eta}\right]^{\frac{1}{2}}d\eta. \tag{48}$$

The integral in equation (48) is the imaginary part of

$$\int_0^{\frac{1}{2}\pi} e^{2in\theta}\left[\frac{1-\sin\theta}{\sin\theta}\right]^{\frac{1}{2}}d\theta. \tag{49}$$

The integral may be taken in the complex plane from $Z = 1$ to $Z = i$, where

$$Z = e^{i\theta},\quad d\theta = dZ/iZ \quad\text{and}\quad \sin\theta = (Z^2-1)/2iZ. \tag{50}$$

Substitution of equation (50) into (49) yields the integral

$$\int_1^i \frac{Z^{2n}(1+iZ)}{Z(1-Z^2)^{\frac{1}{2}}}dZ. \tag{51}$$

Since there are no singularities in the quadrant concerned, the path of integration may be taken from 1 to 0, then 0 to i. The integral (51) then becomes

$$\int_0^1 \frac{x^{2n}(1+ix)}{x[1-x^2]^{\frac{1}{2}}}dx + \int_0^1 \frac{(iy)^{2n}(1-y)\,i\,dy}{iy(1-y^2]^{\frac{1}{2}}}. \tag{52}$$

The only imaginary part is the second term of the first integral. Upon letting $x^2 = \beta$, it becomes

$$-\frac{1}{2}\int_0^1 \frac{\beta^{n-\frac{1}{2}}}{(1-\beta)^{\frac{1}{2}}}d\beta = -\frac{1}{2}\frac{\Gamma(n+\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(n+1)}.$$

Thus, the coefficients a_n are given by

$$a_n = \frac{\sigma B\phi_W}{\pi\rho U}\left\{\frac{(-1)^n}{2n}\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}\right\}. \tag{53}$$

Values of a_n are shown in table 1. The stream function is obtained by substitution of equation (45) into equation (43) and integration. The result is

$$\psi_1 = \int_0^\infty G_1\,d\xi = \sum_{n=1}^{\infty}\frac{a_n}{2n}\sin(2ny)\,e^{2nx} \quad \text{for} \quad x < 0;$$

$$\psi_1 = \int_0^x G_2\,d\xi + \int_x^\infty G_1\,d\xi = \sum_{n=1}^{\infty}\frac{a_n}{2n}\sin(2ny)\,(2-e^{-2nx}) \quad \text{for} \quad x > 0. \tag{54}$$

The solution is shown graphically by the streamline pattern in figure 5.

The velocity perturbation is then given by differentiation of equation (54), and substitution of a_n from equation (53). Hence

$$u' = \frac{\sigma B\phi_W}{2\pi\rho U}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}\cos(2ny)\,[1\mp(1-e^{\pm 2nx})], \tag{55}$$

$$v' = -\frac{\sigma B\phi_W}{2\pi\rho U}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}\sin(2ny)\,e^{\pm 2nx}.$$

For $x < 0$, the upper sign, and for $x > 0$, the lower sign is to be used. It is obvious that the change in axial velocity at $x = 0$ is one-half of that at $x = \infty$. When the

n	$\pi\rho U a_n/\sigma B\phi_W$	n	$\pi\rho U a_n/\sigma B\phi_W$
1	−0·250000	11	−0·007645
2	0·092500	12	0·006716
3	−0·052080	13	−0·005961
4	0·034180	14	0·005337
5	−0·024610	15	−0·004815
6	0·018800	16	0·004373
7	−0·014960	17	−0·003995
8	0·012270	18	0·003668
9	−0·010300	19	−0·003384
10	0·008809	20	0·003134

TABLE 1. Values of the Fourier coefficients a_n.

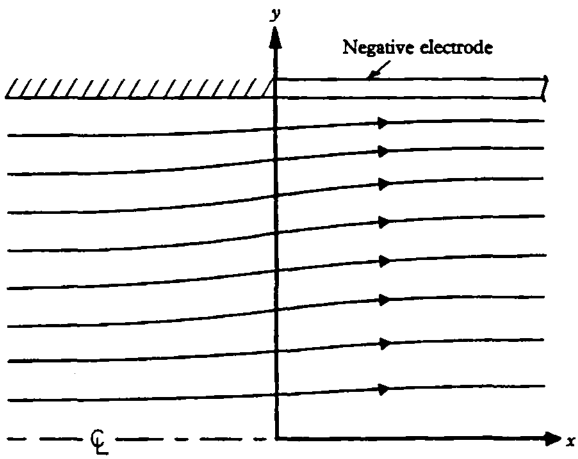

FIGURE 5. Streamline pattern in the upper half of the channel.

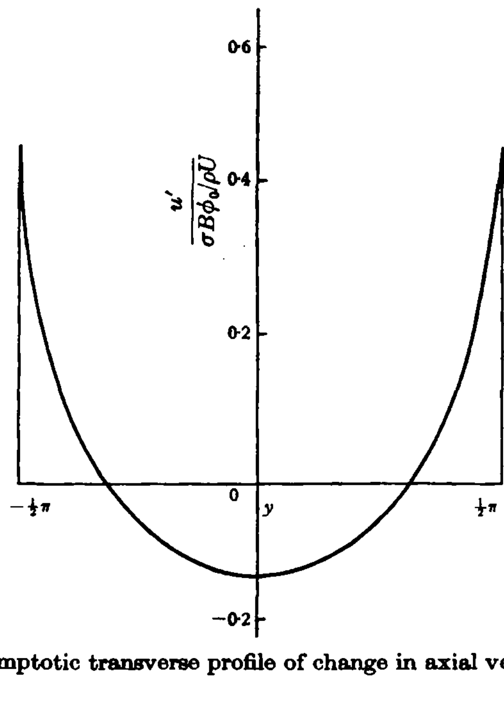

FIGURE 6. Asymptotic transverse profile of change in axial velocity, as $x \to \infty$.

fluid leaves the magnetic field, the effects are exactly doubled and the velocity change is also doubled.

It can be seen from the form of the solution that the changes in velocity are linearly dependent on a modified magnetic interaction parameter $I' = \sigma B \phi_W / \rho U^2$, or in terms of I,

$$I' = (\phi_W / BUh) I. \tag{56}$$

The maximum value of u' occurs at the wall; this is given by $u'/U = 0{\cdot}44 I'$ as $x \to \infty$. Thus, the linearization of the flow is consistent with the assumption of small magnetic interaction parameter. Also, if the two electrodes are at the same potential, the $u' = v' = 0$. The axial velocity perturbation profile for $x \to \infty$ is shown in figure 6 and the development is shown in figure 7.

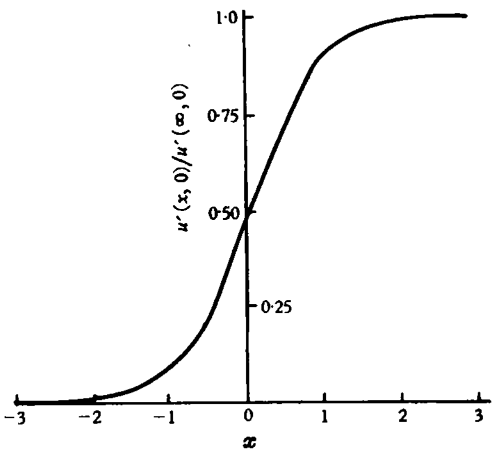

FIGURE 7. Axial variation of change in axial velocity along axis of channel.

5. Asymptotic velocity profile

In the region of the channel far downstream from the entrance the velocity can be calculated directly, without first finding the solution throughout the entire channel.

As x becomes very large, the vertical component of the velocity becomes very small. Therefore, the vorticity is

$$\zeta = -\partial u / \partial y \tag{57}$$

so that

$$u = -\int \zeta \, dy. \tag{58}$$

From equation (9), the vorticity is given by

$$\zeta = \frac{\sigma B \phi_W}{\pi \rho U} \left[\frac{1 - \sin \eta}{\sin \eta} \right]^{\frac{1}{2}}, \tag{59}$$

and integration of the radical with respect to η yields

$$2 \sinh^{-1} (\sin \eta)^{\frac{1}{2}} + \text{const.} \tag{60}$$

Since the average velocity must be U, the constant may be evaluated, from which it is found that

$$u' = \frac{2 \sigma B \phi_W}{\pi \rho U} [\ln 2 - \sinh^{-1} (\cos y)^{\frac{1}{2}}]. \tag{61}$$

Thus $$u'/U = -0{\cdot}118\,\sigma B\phi_W/\rho U^2 \quad \text{at} \quad y = 0 \tag{62}$$

and $$u'/U = 0{\cdot}442\,\sigma B\phi_W/\rho U^2 \quad \text{at the walls.} \tag{63}$$

For $I' = \frac{1}{4}$, the velocity variation at the wall is about 10 % of the initial velocity. For larger values of the parameter, the assumption of small velocity perturbations is no longer valid.

This solution is qualitatively similar to that found by Shercliff (1956) for the flow distortion at the inlet to a flowmeter; however, equation (18) predicts that the flow disturbance would be about 50 % greater than that for the flowmeter problem. Also, Shercliff gives only the asymptotic velocity profile and not the development of the perturbation, as given by equation (55).

6. Discussion

Measurements of the flow distortion have recently been made in copper sulphate solutions by Rossow, Jones & Huerta (1961). The mean flow velocity was 4·22 in./sec, taken at the point where $u' = 0$ from the theory, which corresponds to 1·17 in. from the centre-line. The value of $u'(0)$ was then 0·83 in./sec. Using a magnetic field of 4000 G, equation (62) predicts that $u' = 1{\cdot}15$ in./sec., which is higher by about 28 %. This is good agreement, considering that viscosity was neglected, and that the magnetic field does not abruptly decrease to zero at $x = 0$.

This work was supported by the United States Air Force Office of Scientific Research, under contract 49(638)–914. The authors gratefully acknowledge the helpful suggestions of Dr H. Hurwitz, Jr.

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